ON NONUNIFORM ASYMPTOTIC STABILITY

(K NERAVNOMERNOI ASIMPTOTICHESKOI USTOICHIVOSTI)

PMM Vol.27, No.2, 1963, pp. 231-243

M. REGISH
(Timishoara, Roumanian People's Republic)

(Received January 12, 1961)

Consider "nonuniform" (relative to the initial value of the parameter t) asymptotic stability, in the first approximation. It is shown that there are various types of conditions leading to nonuniform behavior with respect to t_0 of the solutions of a linear system of differential equations, and the relationships between these various conditions are developed. Criteria for stability in the first approximation are introduced, criteria which contain a group of "uniform" equivalent criteria of Persidskii [1, 2], Malkin [3] and Perron [4], and also the "generalized" criteria of Liapunov [5] (see [6, p.72], [7, p.364, Note 2] and Malkin [8], see also [6, p.369]). It is shown that the last mentioned criteria contain as a special case the generalized criteria of Liapunov. The criteria [9], considered for the case of ordinary (i.e. unconditional) stability, are also special cases of the results of the present paper.

In this paper certain ideas of Krein [10] (see also [11]) are employed; and also, following Bellman [12], a theorem of Banach-Steinhaus [13, 14] is employed. The paper admits of further generalization in the direction of the treatment of the equations considered, as well as in the direction of considering the problem in arbitrary Banach spaces, and in the discussion of the so-called conditional stability (i.e. dichotomy) in the sense of the paper of Massera and Schaffer [15].

I take this occasion to thank A. Khalana for his many suggestions, which have influenced the paper, as well as for his aid, which contributed to the completion of the paper.

1. Preliminary remarks. Let R_n be a n-dimensional space and $R_n{}^*$ be the space of all square n by n matrices. The norm of a vector x and of a matrix A will be denoted, respectively, by

$$\|x\| = \|(x_i)_{1\leqslant i\leqslant n}\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \qquad \|A\| = \|(a_{ij})_{1\leqslant i,\ j\leqslant n}\| = \max_{1\leqslant i\leqslant n}\left(\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}$$

We shall consider the following differential equations:

$$\frac{dx}{dt} = A(t)x \qquad (t \geqslant 0) \tag{1.1}$$

$$\frac{dx}{dt} = A(t)x + \varphi(t) \qquad (t \geqslant 0) \tag{1.2}$$

$$\frac{dX}{dt} = A(t)X \qquad (t \geqslant 0) \tag{1.3}$$

where $\varphi(t)$ is a vector-valued function defined on $[0, +\infty)$ and having values in the set R_n, and which is subject in the following to various conditions; $A(t)$ is a function defined on $[0, +\infty)$ and with values in R_n^*, and such that there exists a number $A_0 > 0$ such that $\|A(t)\| \leqslant A_0$ on $[0, +\infty)$; the matrix X is an n by n square matrix.

The solution $x(t)$ of equation (1.1) or (1.2), for which $x(t_0) = x_0$, will be denoted by $x(t, t_0, x_0)$. Analogously, the solution $X(t)$ of the equation (1.3), for which $X(t_0) = X_0$, will be denoted by $X(t, t_0, X_0)$.

If I is the identity matrix of R_n^*, then, introducing the usual notation: $X(t, t_0) = X(t, t_0, I)$, we have the following relations:

$$X(t,t_0,X_0) = X(t,t_0)X_0, \qquad X(t,t_0) = X(t,t_1)X(t_1,t_0), \qquad X(t_1,t_2) = X^{-1}(t_2,t_1)$$
$$t,\ t_0\ t_1,\ t_2 \geqslant 0$$

If $x(t, t_0, x_0)$ is a solution of equation (1.1), then

$$x(t, t_0, x_0) = X(t, t_0)x_0 \tag{1.4}$$

If $x(t, t_0, x_0)$ is a solution of (1.2), then

$$x(t, t_0, x_0) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)\varphi(s)\,ds \tag{1.5}$$

Next we shall define a class of function spaces with which we shall have to deal frequently in the sequel.

Let L be the set of all measurable functions $\varphi(t)$ on $[0, +\infty)$, with values in R_n, and integrable on each finite interval, and suppose that a is a real number. Let us introduce the following Banach function spaces (assuming that one identifies all equivalent* functions):

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* **Equivalent functions are functions which coincide up to a set of measure zero.**

$$L_a{}^p = \left\{\varphi : \varphi \in L, \int_0^{+\infty} \|\varphi(s)\|^p e^{pas} ds < +\infty\right\}, \qquad p \in [1, +\infty) \qquad (1.6)$$

that is, the set of all functions φ in L (i.e. measurable and satisfying the inequality between braces) and

$$L_a{}^\infty = \{\varphi : \varphi \in L, \operatorname{ess\,sup}_{s \geqslant 0} \|\varphi(s)\| e^{as} < +\infty\} \qquad (1.7)$$

that is, the set of functions in L which satisfy the inequality between curly braces*. In these spaces the norm is defined in the usual manner

$$\|\varphi\|_{L_a{}^p} = \|\varphi\|_{(p,a)} = \left[\int_0^{+\infty} \|\varphi(s)\|^p e^{pas} ds\right]^{1/p}, \qquad p \in [1, +\infty) \quad (1.8)$$

$$\|\varphi\|_{L_a{}^\infty} = \|\varphi\|_{(\infty, a)} = \operatorname{ess\,sup}_{s \geqslant 0} \|\varphi(s)\| e^{as} \qquad (1.9)$$

Note 1.1. If a and b are two real numbers, then the linear operator $\Omega_a{}^b$ which maps elements $\varphi(s)$ in $L_a{}^p$ into elements $\varphi^*(s)$ in $L_b{}^p$, defined as follows:

$$\varphi^*(s) = \Omega_a{}^b \varphi = \varphi(s) e^{(a-b)s} \qquad (1.10)$$

possesses an inverse operator $[\Omega_a{}^b]^{-1} = \Omega_b{}^a$ and is an isomorphic and isometric (that is, preserves the distance between elements) map of the space $L_a{}^p$ on the space $L_b{}^p$.

In particular, all spaces $L_a{}^p$ are isometric to the space $L_0{}^p = L^p$. Further, if $a < b$, then $L_a{}^p$ contains $L_b{}^p$, and the topology of the space $L_b{}^p$ is stronger than the topology which is induced in $L_b{}^p$ by the topology of $L_a{}^p$ (in other words, from the convergence of a sequence of functions in the sense of the norm of the space $L_b{}^p$ it necessarily follows that the same sequence converges in the sense of the norm in the space $L_a{}^p$; in this sense, employing the terminology of [15,16], we shall say that the space $L_b{}^p$ is stronger than the space $L_a{}^p$).

2. Investigation of the systems (1.1), (1.2) and (1.3). Let us now pass to the consideration of a series of conditions to be imposed on the solutions of the equations (1.2) and (1.3). These conditions generalize the restrictions which are imposed on the motions of

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* The symbol ess $\sup_{s \geqslant 0} y(s)$ denotes the lower bound of all numbers which are upper bounds of the function $y(s)$ on sets which differ from the semi axis $s \geqslant 0$ by sets of measure zero.

systems of first approximation in the above mentioned criteria.

$A\ (h,\ a,\ b)$. There exist $h,\ a,\ b > 0$ such that

$$\|X(t, s)\| \leqslant he^{as}e^{-bt} \qquad (t \geqslant s \geqslant 0)$$

$B\ (a,\ b,\ p)$. There exist $a,\ b > 0$ and p in $[1,\ +\infty]$ such that

$$\sup_{t\geqslant 0}\int_0^t e^{pbt}\|X(t, s)\|^p e^{-pas}\,ds < +\infty$$

$C\ (a,\ b,\ p)$. There exist $a,\ b > 0$ and p in $[1,\ +\infty]$ such that the solution $x(t,\ t_0,\ x_0)$ of equation (1.2) belongs to L_b^{∞} for each φ in L^p, that is

$$\text{ess sup}\,\|x(t, t_0, x_0)\|e^{bt} = N < \infty \quad \text{for} \quad \int_0^{\infty}\|\varphi(s)\|^p e^{pas}\,ds < \infty$$

The fundamental result of this section is contained in the following theorem:

Theorem 2.1. The following relations are valid:

Equivalences

$$\begin{array}{ll}(\alpha) & A(h, a, b) \Leftrightarrow C(a, b, 1) \\ (\beta) & B(a, b, q) \Leftrightarrow C(a, b, p)\end{array} \qquad \text{for } \frac{1}{p}+\frac{1}{q}=1 \qquad \begin{cases}p \in (1, +\infty] \\ q \in [1, +\infty)\end{cases}$$

Implications

(γ) $B(a, b, q),\ C(a, b, p) \Rightarrow A(h, a, b)$ for $p \in (1, +\infty],\ q \in [1, +\infty)$

(δ) $A(h, a, b) \Rightarrow C(a, b-\varepsilon, p)$ for $p \in (1, +\infty]$ and arbitrarily given $\varepsilon > 0$

Before proving this theorem, we shall formulate two lemmas, whose proof will be omitted, for the sake of brevity, in view of their simplicity. We only note that one employs an inequality of Bellman [17].

Lemma 2.1. There exists $T > 0$ such that

$$\|X(t, s) - X(t, t_0)\| \leqslant \tfrac{1}{2}\|X(t, t_0)\| \text{ for } s \in [t_0 - T,\ t_0 + T] \qquad (t,\ t_0 \geqslant 0)$$

The following lemma is an immediate consequence of the lemma just proved:

Lemma 2.2. There exists $T>0$ such that

$$\frac{1}{2}\|X(t,t_0)\|\leqslant\|X(t,s)\|\leqslant\frac{3}{2}\|X(t,t_0)\| \quad \text{for } s\in[t_0-T,t_0+T] \quad (t,t_0\geqslant 0) \tag{2.1}$$

Let us now formulate the important theorem of Banach-Steinhaus type on which is based, in an essential way, the proof of Theorem 2.1.

Theorem 2.2. Suppose that, for certain real numbers a and b and a certain p in $[1, +\infty]$, the integral

$$V_t u=\int_0^t X(t,s)\,u(s)\,ds \qquad (t\geqslant 0)$$

defines a linear operator V

$$V: L_a{}^p\to L_b{}^\infty, \qquad (Vu)(t)=V_t u$$

Then this operator is a continuous operator, that is to say, there exists a number $M>0$ such that

$$\|Vu\|_{(\infty,b)}\leqslant M\|u\|_{(p,a)}, \qquad u\in L_a^p \tag{2.2}$$

Furthermore,

a) if p is in $(1, +\infty]$, then

$$\sup_{t\geqslant 0}\left[\int_0^t e^{qbt}\|X(t,s)\|^q e^{-qas}\,ds\right]^{\frac{1}{q}}<+\infty \qquad \left(q=\frac{p}{p-1}\right)$$

b) if $p=1$, then

$$\sup_{t\geqslant s\geqslant 0} e^{bt}\|X(t,s)\|e^{-as}<+\infty$$

Proof. Let us prove first the first part of the theorem.

Suppose that $\{t_k\}$ is the sequence of all rational numbers, arranged in a certain order, and suppose that the linear bounded operators

$$V_k: L_a{}^p\to R_n \qquad (k=1,2,\ldots)$$

are defined by the formulas

$$V_k u=e^{bt_k}V_{t_k}u \qquad (k=1,2,\ldots)$$

Since, for each u in $L_a{}^p$ we have, by hypothesis, that

$$\sup_{k\geqslant 1} \|V_k u\| < +\infty$$

then, according to the Banach-Steinhaus theorem [13] it follows that there exists a positive number M for which

$$\|V_k\| \leqslant M \qquad (k = 1, 2, \ldots)$$

Consequently

$$\|V_k u\| \leqslant M \|u\|_{(p,\,a)} \quad \text{or} \quad e^{bt_k} \left\| \int_0^{t_k} X(t_k, s)\, u(s)\, ds \right\| \leqslant M \|u\|_{(p,\,a)} \quad (k = 1, 2, \ldots)$$

which, by continuity, implies that

$$e^{bt} \left\| \int_0^t X(t, s)\, u(s)\, ds \right\| \leqslant M \|u\|_{(p,\,a)} \quad (t \geqslant 0) \quad \text{or} \quad \|Vu\|_{(\infty,\,b)} \leqslant M \|u\|_{(p,\,a)} \tag{2.3}$$

Let us now prove the second part of the theorem.

Suppose that $x_{1j}(t, s), \ldots, x_{nj}(t, s)$ $(j = 1, \ldots, n)$ are the columns of the matrix $X(t, s)$.

a) Let p be in $(1, +\infty)$, and let $t \geqslant 0$ be arbitrary, but fixed during the discussion. Consider the functions

$$u_i(s) = (u_{ij}(s))_{1\leqslant j\leqslant n} \qquad (i = 1, \ldots, n)$$

where

$$u_{ij}(s) = \|X(t, s)\|^{\frac{q}{p}-1} e^{-qas} x_{ij}(t, s) \left[\int_0^t \|X(t, s)\|^q e^{-qas}\, ds\right]^{-\frac{1}{p}}, \qquad s \in [0, t]$$

$$u_{ij}(s) = 0, \qquad s > t \qquad (i, j = 1, \ldots, n)$$

It is easily seen then that

$$u_i \in L_a^p, \qquad \|u_i\|_{(p,\,a)} \leqslant 1 \qquad (i = 1, \ldots, n)$$

Applying inequality (2.3) we then obtain that

$$Me^{-bt} \geqslant \left\| \int_0^t X(t, s)\, u_i(s)\, ds \right\| = \left\| \int_0^t \left[\sum_{\alpha=1}^n \left(\sum_{j=1}^n x_{\alpha j}(t, s)\, u_{ij}(s)\right) e_\alpha\right] ds \right\| =$$

$$= \left\{\sum_{\alpha=1}^n \left(\int_0^t \left[\sum_{j=1}^n x_{\alpha j}(t, s)\, u_{ij}(s)\right] ds\right)^2\right\}^{\frac{1}{2}} \geqslant \left| \int_0^t \left[\sum_{j=1}^n x_{ij}(t, s)\, u_{ij}(s)\right] ds \right| =$$

$$= \int_0^t \left[\sum_{j=1}^n x_{ij}^2(t,s)\right] \| X(t,s)\|^{\frac{q}{p}-1} e^{-qas} \left[\int_0^t \| X(t,s)\|^q e^{-qas}\, ds\right]^{-\frac{1}{p}} ds \qquad (i = 1, \ldots, n)$$

from which it follows that

$$Me^{-bt} \geqslant \max_{1 \leqslant i \leqslant n} \int_0^t \left[\sum_{j=1}^n x_{ij}^2(t,s)\right] \| X(t,s)\|^{\frac{q}{p}-1} e^{-qas} \left[\int_0^t \| X(t,s)\|^q e^{-qas}\, ds\right]^{-\frac{1}{p}} ds$$

Now, as is easily seen, for arbitrary integrable non-negative functions on the set E, $f_1(s)$, ..., $f_n(s)$, one has the inequality

$$\max_{1 \leqslant i \leqslant n} \int_E f_i(s)\, ds \geqslant \frac{1}{n} \int_E [\max_{1 \leqslant i \leqslant n} f_i(s)]\, ds$$

and hence one obtains finally that

$$Me^{-bt} \geqslant \frac{1}{n} \int_0^t \max_{1 \leqslant i \leqslant n} \left[\sum_{j=1}^n x_{ij}^2(t,s)\right] \| X(t,s)\|^{\frac{q}{p}-1} e^{-qas} \left[\int_0^t \| X(t,s)\|^q e^{-qas}\, ds\right]^{-\frac{1}{p}} ds =$$

$$= \frac{1}{n} \left[\int_0^t \| X(t,s)\|^q e^{-qas}\, ds\right]^{\frac{1}{q}}$$

as was desired.

The case $p = +\infty$ may be treated analogously, the difference being that the functions $u_{ij}(s)$ are then defined more simply

$$u_{ij}(s) = \frac{x_{ij}(t,s)\, e^{-as}}{\| X(t,s)\|}, \quad s \in [0, t]; \quad u_{ij}(s) = 0, \quad s > t \qquad (i, j = 1, \ldots, n)$$

b) Suppose that $p = 1$ and $t \geqslant t_0 \geqslant 0$ is an arbitrary number, fixed during the discussion. Applying Lemma 2.2 to the equation

$$\frac{dY}{d\tau} = -YA(\tau)$$

that is, to the matrix $Y(\tau, \sigma) = X^{-1}(\tau, \sigma) = X(\sigma, \tau)$, one can conclude the existence of a number T, independent of t and s, such that

$$\| X(s, t_0)\| \leqslant \tfrac{3}{2} \| X(t_0, t_0)\| = \tfrac{3}{2} < 2 \quad \text{for } s \in [t_0 - T,\ t_0 + T], \qquad t_0 \geqslant 0$$

Consider now the function

$$u(s)=\begin{cases}(e^{-as}/4T)\,X(s,t_0)\,x_0, & s\in[\alpha,\beta]\\ 0, & 0\leqslant s\notin[\alpha,\beta]\end{cases}\qquad \begin{pmatrix}\alpha=\max(0,t_0-T)\\ \beta=\min(t,t_0+T)\end{pmatrix}$$

It is easily seen that

$$u\in L_a^1,\qquad \|u\|_{(1,a)}\leqslant\|x_0\|$$

Applying the inequality (2.3), we obtain that

$$Me^{-bt}\geqslant\left\|\int_0^t X(t,s)\,u(s)\,ds\right\|=\frac{1}{4T}\left\|\int_\alpha^\beta X(t,s)\,X(s,t_0)\,x_0e^{-as}\,ds\right\|=$$

$$=\frac{1}{4T}\|X(t,t_0)\,x_0\|\int_\alpha^\beta e^{-as}\,ds\geqslant\frac{1}{4T}\|X(t,t_0)\,x_0\|\,e^{-a\beta}\,[\beta-\alpha]$$

and, consequently, in view of the arbitrariness of x_0, we obtain that

$$\|X(t,t_0)\|\leqslant 4Me^{aT}e^{at_0}e^{-bt},\qquad t\geqslant T,\quad \|X(t,t_0)\|\leqslant 2,\qquad t<T$$

Introducing the notation

$$h=\max[2/\min(e^{at_0}e^{-bt});\ 4Me^{2aT}]\ (0\leqslant t\leqslant t_0\leqslant T),$$

we obtain finally that

$$\|X(t,t_0)\|\leqslant he^{at_0}e^{-bt},\qquad t\geqslant t_0\geqslant 0$$

and Theorem 2.2 is proved.

Note 2.1. In case (a) the boundedness of the matrix $A(t)$ for $t\geqslant 0$ turns out to be unessential, while in case (b), on the contrary, it is extremely essential, in order that the proof of Theorem 2.2 carry over without change to an arbitrary Banach space, instead of R_n.

Note 2.2. A similar theorem appears, without proof, in Bellman's paper [12]. There the theorem is formulated for a finite dimensional space, with $a = b = 0$, $p = 1, 2, \ldots, +\infty$ (see also, in this connection, [13, 14]).

Proof of Theorem 2.1. The implication $A(h, a, b)\Rightarrow C(a, b, 1)$ is obvious. In order to prove that $C(a, b, 1)\Rightarrow A(h, a, b)$, consider the solution

$$x(t,0,0)=\int_0^t X(t,s)\,\varphi(s)\,ds\qquad (t\geqslant 0)\tag{2.4}$$

of equation (1.2). In view of the hypotheses, the function

$$(V_t\varphi)\, e^{bt} = xe^{bt}$$

is a bounded function for $t \geqslant 0$ for each φ in L_a^1. According to Theorem 2.2, from this follows the existence of a number $h > 0$ such that

$$e^{-as} \| X(t, s) \| e^{bt} \leqslant h, \qquad t \geqslant s \geqslant 0 \tag{2.5}$$

and this implies the condition $A(h, a, b)$. The equivalence of the conditions $C(a, b, p)$ and $B(a, b, q)$ can be proved analogously.

Let us prove next that $B(a, b, q) \Rightarrow A(h, a, b)$. In order to do this, let us suppose that $B(a, b, q)$ holds, while $A(h, a, b)$ does not, that is to say that there does not exist a number $h > 0$ for which inequality (2.5) is valid.

Let us determine a sequence $\{h_n\}$ such that

$$\lim h_n = +\infty \quad \text{as } n \to +\infty \tag{2.6}$$

Then there exist sequences $\{t_n\}$, $\{s_n\}$ such that

$$t_n \geqslant s_n, \qquad e^{-as_n} \| X(t_n, s_n) \| e^{bt_n} > h_n \qquad (n = 1, 2, 3, \ldots) \tag{2.7}$$

It is easy to see that necessarily $\sup_n t_n = +\infty$; indeed, if this were not the case, then (2.6) and (2.7) would hold on a compact set*

$$\{(t, s) : 0 \leqslant s \leqslant t \leqslant \sup_n t_n < +\infty\}$$

which is impossible, since $X(t, s)$ is continuous in each argument, and

$$\| X(t, s) \| \leqslant \| X(t, 0) \| \| X(0, s) \|$$

Extracting (if need be) three subsequences, and labeling them, respectively, $\{h_n\}$, $\{t_n\}$, $\{s_n\}$, it may be supposed further that $\lim t_n = +\infty$ as $n \to +\infty$.

We now consider two cases.

a) Suppose that $\sup_n s_n < +\infty$; in this case, in view of the condition $B(a, b, q)$, there exists a constant $C > 0$ such that

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* That is, on a set such that every bounded subsequence of it possesses at least one limit element.

$$C \geqslant \int_0^{t_n} e^{-qas} \| X(t_n, s) \|^q e^{qbt_n} ds \geqslant \int_{s_n}^{s_n+T} e^{-qas} \| X(t_n, s) \|^q e^{qbt_n} ds$$

for all numbers n which are sufficiently large, where T is the number which is determined by means of Lemma 2.2. According to this lemma, and inequality (2.7), one may assert further that

$$C \geqslant \frac{1}{2^q} \int_{s_n}^{s_n+T} e^{-qas} \| X(t_n, s_n) \|^q e^{qbt_n} ds \geqslant \frac{T}{2^q} e^{-qaT} h_n^q \to +\infty$$

as $n \to +\infty$, which is impossible.

b) Suppose now that $\sup_n s_n = +\infty$; in this case we may assume that $\lim s_n = +\infty$ as $n \to \infty$, and then, as in case (a), we obtain the inequality

$$C \geqslant \int_0^{t_n} e^{-qas} \| X(t_n, s) \|^q e^{qbt_n} ds > \int_{s_n-T}^{s_n} e^{-qas} \| X(t_n, s) \|^q e^{qbt_n} ds \geqslant \frac{T}{2^q} h_n^q \to +\infty$$

as $n \to +\infty$, which is also impossible.

Finally, the implication $A(h, a, b) \Rightarrow C(a, b-\varepsilon, p)$ for arbitrary p in $(1, +\infty]$ and arbitrary $\varepsilon > 0$ follows in a corresponding manner from the inequality

$$\| x(t, 0, 0) \| e^{(b-\varepsilon)t} \leqslant e^{(b-\varepsilon)t} \int_0^t \| X(t, s) \| \cdot \| \varphi(s) \| ds \leqslant$$

$$\leqslant e^{(b-\varepsilon)t} \left(\int_0^t \| X(t, s) \|^q e^{-qas} ds \right)^{\frac{1}{q}} \left(\int_0^t \| \varphi(s) \|^p e^{pas} ds \right)^{\frac{1}{p}} \leqslant h t^{\frac{1}{q}} e^{-\varepsilon t} \| \varphi \|_{(p,\, a)} \leqslant C_1$$

or from the inequality

$$\| x(t, 0, 0) \| e^{(b-\varepsilon)t} \leqslant (\sup_{s \geqslant 0} \| \varphi(s) \| e^{as}) e^{(b-\varepsilon)t} \int_0^t \| X(t, s) \| e^{-as} ds \leqslant h t e^{-\varepsilon t} \| \varphi \|_{(\infty,\, a)} \leqslant C_2$$

where C_1, $C_2 > 0$ are sufficiently large constants. The theorem is thus completely proved.

3. Application to the problem of stability to the first approximation. Consider the equation

$$\frac{dx}{dt} = A(t) x + \Phi(x, t), \qquad x \in R_n \tag{3.1}$$

where $A(t)$ is the matrix-valued function of (1.1), and $\Phi(x, t)$ is a vector function, defined and continuous on the set

$$t \geqslant 0, \quad \|x\| \leqslant D \qquad (D > 0) \tag{3.2}$$

and satisfying the condition

$$\|\Phi(x, t)\| \leqslant f(t) \|x\|^m \qquad (m \geqslant 1) \tag{3.3}$$

where the number m and the real valued function $f(t)$ are specified more precisely below.

For equation (3.1) there exist various important criteria of stability in the first approximation; they may be divided into two classes. In the first class one finds the criteria based on the condition imposed on the matrix $X(t, t_0)$ by Persidskii [1]

$$\|X(t, t_0)\| \leqslant he^{-\alpha(t-t_0)}, \qquad t \geqslant t_0 \geqslant 0 \tag{3.4}$$

These criteria constitute the class of "uniform" criteria. In the second class one finds the "generalized" criteria [5,8,9,18], that is to say, the extensions of the criteria of the first class to the "non-uniform" case. For example, the condition of Malkin, in the present notation, has the form

$$\|X(t, t_0)\| \leqslant he^{\beta t_0} e^{-\alpha(t-t_0)}, \qquad m > \frac{\alpha+\beta}{\alpha} \tag{3.5}$$

Let us now formulate certain criteria of stability in the first approximation which comprise all these criteria just mentioned.

Theorem 3.1. Suppose that the equation of first approximation (1.1) satisfies condition $A(h, a, b)$ for some $h > 0$, $a \geqslant b > 0$, and that $\Phi(x, t)$ satisfies the inequality (3.3); and consider the following cases:

$$(\alpha) \quad m > \frac{a}{b}, \quad a \geqslant b, \quad p \in [1, +\infty], \quad f \in L_0^p, \quad \|f\|_{(p,0)} \leqslant K_p$$

$$\|x_0\| \leqslant D_1, \quad \|x_0\| \leqslant [2(m-1)h^m K_p \| e^{(a-mb)s}(\| q, 0)]^{-\frac{1}{m-1}} e^{-\frac{(a-b)m}{m-1} t_0}$$

$$(\beta) \quad m = \frac{a}{b}, \quad a > b, \quad p = 1, \quad f \in L_0^{\cdot 1}, \quad \|f\|_{(1,0)} \leqslant K_1$$

$$\|x_0\| \leqslant D_1, \quad \|x_0\| \leqslant [2(m-1)h^m K_1]^{-\frac{1}{m-1}} e^{-\frac{(a-b)m}{m-1} t_0}$$

$$(\gamma) \quad m = \frac{a}{b}, \quad a = b, \quad p \in [1, +\infty], \quad f \in L_0^p, \quad \|f\|_{(p,0)} \leqslant K_p$$

$$\|x_0\| \leqslant D_1$$

$$(\delta)\qquad m=\frac{a}{b},\quad a=b,\quad p=+\infty,\quad f\in L_0^{\infty},\quad \|f\|_{(\infty,0)}\leqslant K_\infty$$
$$K_\infty<\frac{b}{h},\quad \|x_0\|<D_1$$

where $q=\frac{p}{p-1}$, $D_1<D$, K_p, $p\in[1,+\infty]$ are constants. Then every solution $x(t,\ t_0,\ x_0)$ of equation (3.1) satisfies the implications

$$(\alpha),\ (\beta),\ (\gamma)\Rightarrow\|x(t,t_0,x_0)\|\leqslant He^{at_0}e^{-bt}\|x_0\|,\quad t\geqslant t_0\geqslant 0$$
$$(\delta)\Rightarrow\|x(t,t_0,x_0)\|\leqslant he^{ct_0}e^{-ct}\|x_0\|,\quad t\geqslant t_0\geqslant 0$$

where

$H=2^{\frac{1}{m-1}}h$ in cases (α), (β); $H=he^{hK_p}$ in case (γ); $c=b-hK_\infty>0$ in case (δ)

Proof of Theorem 3.1. We have that

$$\|x(t,t_0,x_0)\|\leqslant\|X(t,t_0)\|\,\|x_0\|+\int_{t_0}^{t}\|X(t,\tau)\|\,\|\Phi[x(\tau),\tau]\|\,d\tau,\qquad t\geqslant t_0\geqslant 0$$

Setting

$$t=t_0+s,\qquad \tau=t_0+\sigma,\qquad \|x(t_0+s,\ t_0,\ x_0)\|=\varphi(s),\qquad \varphi(0)=\|x_0\|$$

and observing the condition $A(h,\ a,\ b)$, we obtain that

$$\varphi(s)\leqslant he^{at_0}e^{-b(t_0+s)}\varphi(0)+\int_0^s he^{a(t_0+\sigma)}e^{-b(t_0+s)}f(t_0+\sigma)\,\varphi^m(\sigma)\,d\sigma$$

which means that

$$\varphi(s)\leqslant ke^{-bs}\varphi(0)+ke^{-bs}\int_0^s e^{a\sigma}f(t_0+\sigma)\,\varphi^m(\sigma)\,d\sigma,\qquad k=\begin{Bmatrix}he^{(a-b)t_0}, & a>b\\ h, & a\leqslant b\end{Bmatrix},\qquad h>1$$

It is readily seen that if $\psi(s)$ is defined by the equation

$$\psi(s)=ke^{-bs}\varphi(0)+ke^{-bs}\int_0^s e^{a\sigma}f(t_0+\sigma)\,\psi^m(\sigma)\,d\sigma,\qquad s\geqslant 0\tag{3.6}$$

then, on the one hand

$$\varphi(s)\leqslant\psi(s),\qquad s\geqslant 0$$

while, on the other hand, a simple differentiation of (3.6) leads to

$$\dot\psi+b\psi=Q(s)\,\psi^m,\qquad Q(s)=ke^{(a-b)s}f(t_0+s)\tag{3.7}$$

In cases (α), (β) this equation is of Bernoulli type. Solving it, we obtain

$$\psi(s)=\left[[\psi(0)]^{1-m}e^{(m-1)bs}-(m-1)ke^{(m-1)bs}\int_0^s e^{(a-mb)\sigma}f(t_0+\sigma)\,d\sigma\right]^{-\frac{1}{m-1}},\quad s\geqslant 0$$

But $\psi(0) = k\varphi(0)$ and $\varphi(s) \leqslant \psi(s)$, consequently

$$\varphi(s)\leqslant ke^{-bs}\varphi(0)\left\{1-(m-1)k^m[\varphi(0)]^{m-1}\int_0^s e^{(a-mb)\sigma}f(t_0+\sigma)\,d\sigma\right\}^{-\frac{1}{m-1}} \tag{3.8}$$

Case (α). Since in this case $k = he^{(a-b)t_0}$, and

$$\int_0^s f(t_0+\sigma)\,e^{(a-mb)\sigma}\,d\sigma\leqslant\|f\|_{(p,\,0)}\,\|e^{(a-mb)\sigma}\|_{(q,\,0)},\qquad \frac{1}{p}+\frac{1}{q}=1 \tag{3.9}$$

then, from the inequality (α) for $\| x_0 \|$, together with (3.8) and (3.9), it follows that

$$\varphi(s)\leqslant 2^{\frac{1}{m-1}}he^{(a-b)t_0}e^{-bs}\varphi(0)=2^{\frac{1}{m-1}}he^{at_0}e^{-b(t_0+s)}\varphi(0)$$

or, in the preceding notation

$$\|x(t,t_0,x_0)\|\leqslant He^{at_0}e^{-bt}\|x_0\|,\qquad H=2^{\frac{1}{m-1}}h \tag{3.10}$$

Case (β). Since $mb = a$ and $p = 1$, then (3.8) and inequality (β) for $\| x_0 \|$ imply, as before, that (3.10) holds.

Cases (γ) and (δ). In these cases the proof proceeds in an analogous manner, the difference being that equation (3.7) is linear; however, in these cases the results coincide essentially with the results of Krein [10] and Kucher[11].

Theorem 3.2. If equation (1.2) satisfies condition $C(a,\ b,\ r)$, $a,\ b > 0$, r in $[1,\ +\infty]$, then all the conclusions of Theorem 3.1 are valid. Further, in case (β) the following improvement is possible:

(β'). Suppose that for $a > b$, $m \geqslant a/b$

$$\|x_0\|\leqslant[4h]^{-1}e^{-at_0}\left\{4\sup_{\psi}\left\|e^{bt}\int_0^t X(t,s)\,\psi(s)\,ds\right\|_{(\infty,\,0)}\right\}^{\frac{1}{m-1}}$$

where

$$\psi\in L_a^{\,r},\qquad\|\psi\|_{(r,\,a)}\leqslant K_r,\qquad r\in[1,+\infty]\qquad(K_r>0)$$

Then one has

$$\|x(t,t_0,x_0)\|\leqslant H_1e^{at_0}e^{-bt}\|x_0\|,\qquad t\geqslant t_0\geqslant 0,\qquad f\in L_0^{\,r},\qquad \|f\|_{(r,\,0)}\leqslant K_r$$

Proof. Since Theorem 2.1 implies that $C(a, b, r) \Rightarrow A(h, a, b)$, then the first part of the theorem follows automatically. In order to prove the last part, let us suppose that (1.2) satisfies the condition $C(a, b, r)$ for some r in $[1, +\infty]$, $a > b > 0$ and that $m \geqslant a/b$. From $C(a,b,r)$ it follows that

$$\| X(t, t_0) \| \leqslant h e^{at_0} e^{-bt}, \qquad t \geqslant t_0 \geqslant 0$$

Let us choose $h > 1/4$, and let us prove that

$$\| x(t, t_0, x_0) \| < 4h e^{at_0} e^{-bt} \| x_0 \|, \qquad t \geqslant t_0$$

provided that

$$\| x_0 \| \leqslant [4h e^{at_0}]^{-1} \left\{ 4 \sup_\psi \left\| e^{bt} \int_0^t X(t, s) \psi(s)\, ds \right\|_{(\infty, 0)} \right\}^{-\frac{1}{m-1}}$$

Since $\| x(t_0, t_0, x_0) \| = \| x_0 \| < 4h \| x_0 \| e^{(a-b)t_0}$, it follows that, whenever $t \geqslant t_0$, with t sufficiently near t_0, the required inequality does indeed hold. Let us suppose that there exists some number $\tau > t_0$ for which

$$\| x(\tau, t_0, x_0) \| = 4h e^{at_0} e^{-b\tau} \| x_0 \|$$

while, for all t in $[t_0, \tau)$ we have

$$\| x(t, t_0, x_0) \| < 4h e^{at_0} e^{-bt} \| x_0 \| \tag{3.11}$$

This supposition will be shown to lead to a contradiction. Consider the function

$$\varphi(s) = \begin{cases} x(s, t_0, x_0), & s \in [t_0, \tau) \\ 0, & 0 \leqslant s \notin [t_0, \tau) \end{cases}$$

It is readily seen that

$$\psi(s) = [4^m h^m \| x_0 \|^m e^{amt_0}]^{-1} \Phi[\varphi(s), s] \in L_a^r, \qquad \| \psi \|_{(r, a)} \leqslant K_r$$

From this, together with condition $C(a, b, r)$, it follows, for t in $[t_0, \tau)$ that

$$e^{bt} \left\| \int_{t_0}^t X(t, s) \frac{\Phi[x(s), s]}{4^m h^m \| x_0 \|^m e^{amt_0}} ds \right\| \leqslant \left\| e^{bt} \int_0^t X(t, s) \psi(s)\, ds \right\|_{(\infty, 0)}$$

But, for these values of t we have

$$\| x(t, t_0, x_0) \| \leqslant \| X(t, t_0) \| \cdot \| x_0 \| + \left\| \int_{t_0}^t X(t, s) \Phi[x(s), s]\, ds \right\| <$$

$$\leqslant he^{at_0}e^{-bt}\|x_0\| + 4^m h^m e^{amt_0}\|x_0\|^m e^{-bt}\left\| e^{bt}\int_0^t X(t,s)\psi(s)\,ds\right\|$$

that is to say, that

$$\|x(t,t_0,x_0)\| \leqslant he^{at_0}e^{-bt}\|x_0\|\left[1+4\left\|e^{bt}\int_0^t X(t,s)\psi(s)\,ds\right\|_{(\infty,0)}(4he^{at_0})^{m-1}\|x_0\|^{m-}\right]$$

From this, in view of the condition imposed upon $\| x_0 \|$, we obtain finally that

$$\|x(t,t_0,x_0)\| \leqslant 2he^{at_0}e^{-bt}\|x_0\|, \qquad t\in[t_0,\tau)$$

which is impossible, since it contradicts the assumption

$$\|x(\tau,t_0,x_0)\| = 4he^{at_0}e^{-b\tau}\|x_0\|$$

Thus the theorem is proved.

Let us now prove that if equation (1.1) possesses positive eigenvalues, then one always has a certain theorem of stability in the first approximation.

Let us denote by $\lambda_1 \leqslant \ldots \leqslant \lambda_n$ the eigenvalues of (1.1), and by $\mu_1 \geqslant \ldots \geqslant \mu_n$ the eigenvalues of the equation

$$\frac{dY}{d\tau} = -YA(\tau), \qquad Y\in R_n^*$$

Then the following theorem holds:

Theorem 3.3. If $\lambda_1 > 0$, then for arbitrary $a < \mu_n$, $b < \lambda_1$, and a certain $h > 0$, equation (1.1) satisfies condition $A(h, a, b)$ and even condition $C(a, b, p)$ for arbitrary p in $[1, +\infty]$. Consequently, the conclusions of Theorems 3.1 and 3.2 are valid.

The proof of this theorem follows from Lemma 3.1 below.

Let us denote by $N_{ij}(t)$ $(i, j = 1, \ldots, n)$ a normal system of solutions of equation (1.1) (see, for example, [6, pp.323-332]) and let $\lambda_1 \leqslant \ldots \leqslant \lambda_n$ be the corresponding eigenvalues

$$\lambda_j = \text{eigenvalue of } \{N_{1j}(t), \ldots, N_{nj}(t)\} \qquad (j=1,\ldots,n)$$

By $N(t)$ we shall denote the matrix whose columns consist of the normal solutions just mentioned

$$\{N_{1j}(t), \ldots, N_{nj}(t)\}, \qquad t \geqslant 0 \qquad (j = 1, \ldots, n)$$

This matrix possesses the following properties:

a) $N(t) \in R_n^*, \qquad t \geqslant 0$

b) $X(t, t_0) = N(t) N^{-1}(t_0) \qquad (t, t_0 \geqslant 0)$

c) $\dfrac{dN^{-1}(s)}{ds} = -N^{-1}(s) A(s) \qquad (s \geqslant 0)$

d) If the elements of the inverse matrix $N^{-1}(s)$ are denoted by $N_{\alpha\beta}{}^{-1}$, then it follows that $N_{\alpha\beta}{}^{-1}$ $\Delta_{\alpha\beta}/\Delta$, where $\Delta = \det N$, and $\Delta_{\beta\alpha}$ is the algebraic complement of the element $N_{\alpha\beta}$ in the matrix N.

e) If $\mu_1 \geqslant \ldots \geqslant \mu_n$ are eigenvalues of the matrix $N^{-1}(s)$, that is

$$\mu_i = \text{eigenvalue of } \{N_{i1}^{-1}(s), \ldots, N_{in}^{-1}(s)\} \qquad (i = 1, \ldots, n)$$

then one has that (see, for example, [19])

$$0 \geqslant \lambda_\alpha + \mu_\alpha \geqslant -\sigma$$

where $\sigma \geqslant 0$ is the "coefficient of irregularity" of equation (1.1), that is, the number defined by the equation

$$\text{eigenvalue of}\left(\exp - \int_0^t \mathrm{Sp}\, A(u)\, du\right) + \sum_{\alpha=1}^{n} \lambda_\alpha = -\sigma \qquad (\sigma \geqslant 0) \qquad (3.12)$$

Employing this terminology, the following theorem is valid:

Lemma 3.1. If $\lambda < \lambda_1$ and $\mu < \mu_n$, then there exists a number

$$h = h(\mu, \lambda) > 0$$

such that

$$\|X(t, t_0)\| \leqslant h e^{-\mu t_0} e^{-\lambda t} \qquad (t, t_0 \geqslant 0) \qquad (3.13)$$

Proof. Since $\lambda < \lambda_1$ and λ_1 is the least eigenvalue of the matrix $N(t)$, we have that

$$\|N(t)\| \leqslant h_1(\lambda) e^{-\lambda t}, \qquad t \geqslant 0, \qquad h_1(\lambda) \geqslant 0 \qquad (3.14)$$

Similarly

$$\|N^{-1}(t_0)\| \leqslant h_2(\mu) e^{-\mu t_0}, \qquad t_0 \geqslant 0, \qquad h_2(\mu) > 0 \qquad (3.15)$$

From property (*b*), together with (3.14) and (3.15), we then deduce that

$$\|X(t, t_0)\| \leqslant \|N(t)\| \|N^{-1}(t_0)\| \leqslant he^{-\mu t_0} e^{-\lambda t}, \qquad t, t_0 > 0$$

where $h = h_1 h_2$, and the lemma is proved.

Lemma 3.2. For any $\varepsilon > 0$ there exists a number

$$k = k(\varepsilon) > 0$$

such that

$$\|X(t, t_0)\| \leqslant ke^{(\sigma+2\varepsilon) t_0} e^{-(\lambda_1-\varepsilon)(t-t_0)}, \qquad t, t_0 > 0 \tag{3.16}$$

where σ is the number defined by equation (3.12).

Proof. Let us denote the columns of the matrix $X(t, t_0)$ by

$$\{x_{1j}(t, t_0), \ldots, x_{nj}(t, t_0)\} \qquad (j = 1, \ldots, n)$$

From (*b*) we have that

$$x_{ij}(t, t_0) = \sum_{\alpha=1}^{n} N_{i\alpha}(t) N_{\alpha j}^{-1}(t_0) \qquad (i, j = 1, \ldots, n)$$

From this, keeping in mind property (*e*), we obtain that

$$|x_{ij}(t, t_0)| \leqslant \sum_{\alpha=1}^{n} |N_{i\alpha}(t)| \, |N_{\alpha j}^{-1}(t_0)| \leqslant \sum_{\alpha=1}^{n} C_\alpha^{ij} e^{-(\lambda_\alpha-\delta)t} e^{-(\mu_\alpha-\eta)t_0} \leqslant$$

$$\leqslant \sum_{\alpha=1}^{n} C_\alpha^{ij} e^{-(\lambda_\alpha-\delta)(t-t_0)} e^{-(\mu_\alpha+\lambda_\alpha-\delta-\eta)t_0} \leqslant k_{ij} e^{(\sigma+\delta+\eta)t_0} e^{-(\lambda_1-\delta)(t-t_0)}$$

where C_α^{ij}, $k_{ij} = n \max_\alpha C_\alpha^{ij}$ are constants. Consequently, putting $\delta = \eta = \varepsilon > 0$, we obtain finally

$$\|X(t, t_0)\| \leqslant ke^{(\sigma+2\varepsilon)t_0} e^{-(\lambda_1-\varepsilon)(t-t_0)} \qquad (t, t_0 \geqslant 0)$$

as was to be shown.

Note 3.1. The "generalized criteria" of Liapunov follow from the generalized criteria of Malkin ([8], or [6, p.369]), and consequently, they also follow from the criteria expressed in Theorem 3.1. Indeed, since Liapunov's condition means that [7]

$$m > 1 + \frac{\sigma}{\lambda_i} \qquad (i = 1, \ldots, n)$$

where m is the number appearing in Theorem 3.1, λ_i are the eigenvalues of equation (1.1), and σ is the number given by (3.12), then one may choose a sufficiently small $\varepsilon > 0$ such that

$$m > 1 + \frac{\sigma + 2\varepsilon}{\lambda_1 - \varepsilon}$$

Now, since, in view of Lemma 3.2, the conditions of Malkin's criteria are satisfied, this is just what was required to be proved. One may still note that (as may be easily seen from the examples given by Malkin ([6, p.368] or [3])), the criteria of Malkin are not equivalent to Liapunov's criteria.

Note 3.2. The criteria presented in [9], formulated for the case of ordinary (i.e. unconditional) stability, are included in Theorem 3.2 for $m = 2$, $a = 2k$, $b = k$, $p = +\infty$.

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Translated by J.B.D.